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THE SINGULARITIES OF YANG-MILLS CONNECTIONS FOR BUNDLES ON A SURFACE. I. THE LOCAL MODEL

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ABSTRACT. Let Σ be a closed surface, G a compact Lie group, not necessarily connected, with Lie algebra \mathfrak{g} , endowed with an adjoint action invariant scalar product, let $\xi: P \rightarrow \Sigma$ be a principal G -bundle, and pick a Riemannian metric and orientation on Σ , so that the corresponding Yang-Mills equations

$$d_A * K_A = 0$$

are defined, where K_A refers to the curvature of a connection A . For every central Yang-Mills connection A , the data induce a structure of unitary representation of the stabilizer Z_A on the first cohomology group $H_A^1(\Sigma, \text{ad}(\xi))$ with coefficients in the adjoint bundle $\text{ad}(\xi)$, with reference to A , with momentum mapping Θ_A from $H_A^1(\Sigma, \text{ad}(\xi))$ to the dual z_A^* of the Lie algebra \mathfrak{z}_A of Z_A . We show that, for every central Yang-Mills connection A , a suitable Kuranishi map identifies a neighborhood of zero in the Marsden-Weinstein reduced space H_A for Θ_A with a neighborhood of the point $[A]$ in the moduli space of central Yang-Mills connections on ξ .

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Introduction

Let Σ be a closed surface, G a compact Lie group, not necessarily connected, with Lie algebra \mathfrak{g} , and $\xi: P \rightarrow \Sigma$ a principal G -bundle. Further, pick a Riemannian metric on Σ and an *orthogonal structure* on \mathfrak{g} , that is, an adjoint action invariant positive definite inner product. These data then determine a Yang-Mills theory on the space $\mathcal{A}(\xi)$ of connections studied for connected G extensively by ATIYAH-BOTT in [4] to which we refer for background and notation. In particular, the corresponding Yang-Mills equations look like

$$(0.1) \quad d_A * K_A = 0$$

where K_A refers to the curvature of a connection A , and the solutions A of (0.1) are referred to as *Yang-Mills connections*. Let $\mathcal{N}(\xi)$ be the space of *central* Yang-Mills connections on ξ , $\mathcal{G}(\xi)$ the group of gauge transformations, and $N(\xi) = \mathcal{N}(\xi)/\mathcal{G}(\xi)$ the corresponding moduli space; see Section 1 below for a precise definition of a central connection. In the present paper we describe the singularities of $N(\xi)$ explicitly. It turns out that in a suitable sense they are as simple as possible. For reasons that will become clear below we do *not* assume G connected; we have extended the requisite results of ATIYAH-BOTT [4] to bundles with non-connected structure group in [9]. Throughout we shall assume that solutions of (0.1) exist, so that the space $N(\xi)$ is non-empty. For connected structure group this will always be so, cf. [4].

Recall that the *adjoint bundle* $\text{ad}(\xi)$ is associated with ξ via the adjoint representation of G on its Lie algebra \mathfrak{g} ; its sections constitute the Lie algebra $\mathfrak{g}(\xi)$ of infinitesimal gauge transformations of ξ . For a central connection A , not necessarily a Yang-Mills one, the operator $d_A: \Omega^*(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^*(\Sigma, \text{ad}(\xi))$ of covariant derivative is a differential, that is, satisfies $d_A d_A = 0$, even though in general A is *not* flat; see (1.2) below for details. Hence the cohomology groups $H_A^*(\Sigma, \text{ad}(\xi))$ are defined. Moreover the given orthogonal structure on \mathfrak{g} induces a symplectic structure σ_A on the (finite dimensional) vector space $H_A^1(\Sigma, \text{ad}(\xi))$, and the Lie bracket on \mathfrak{g} induces a graded Lie bracket $[\cdot, \cdot]_A$ on $H_A^*(\Sigma, \text{ad}(\xi))$ which, for degree reasons, is *symmetric* on $H_A^1(\Sigma, \text{ad}(\xi))$. Let $Z_A \subseteq \mathcal{G}(\xi)$ be the stabilizer of A . It is a compact Lie group which acts canonically on $H_A^*(\Sigma, \text{ad}(\xi))$, preserving σ_A and $[\cdot, \cdot]_A$, and its Lie algebra \mathfrak{z}_A equals $H_A^0(\Sigma, \text{ad}(\xi))$. The orthogonal structure on \mathfrak{g} induces a canonical isomorphism between $H_A^2(\Sigma, \text{ad}(\xi))$ and the dual \mathfrak{z}_A^* of \mathfrak{z}_A preserving the Z_A -actions. Furthermore, cf. (1.2.5) below, the assignment to $\eta \in H_A^1(\Sigma, \text{ad}(\xi))$ of $\Theta_A(\eta) = \frac{1}{2}[\eta, \eta]_A$ yields a momentum mapping Θ_A for the Z_A -action on the symplectic vector space $H_A^1(\Sigma, \text{ad}(\xi))$, see Section 1 below. MARS DEN-WEINSTEIN reduction [15] yields the space $H_A = \Theta_A^{-1}(0)/Z_A$, and we have the following, cf. (2.32) below for a more precise statement.

Theorem A. *For every central Yang-Mills connection A , a suitable Kuranishi map identifies a neighborhood of the class $[A]$ in $N(\xi)$ with a neighborhood of the class of zero in H_A .*

We shall say that a point $[A]$ of $N(\xi)$ is *non-singular* provided Z_A acts trivially on $H_A^1(\Sigma, \text{ad}(\xi))$. The Theorem entails that $N(\xi)$ is *smooth* near a non-singular point $[A]$ and we have the following immediate consequence.

Corollary B. *The non-singular part of $N(\xi)$ inherits from σ a structure of symplectic manifold.*

In fact, for a central Yang-Mills connection A representing a non-singular point of $N(\xi)$, Theorem A furnishes *Darboux* coordinates on $N(\xi)$ near the class of A ; in a sense, Theorem A or rather (2.32) below yields “Darboux coordinates” near an arbitrary point of $N(\xi)$ where Darboux coordinates now means the whole structure of momentum mapping Θ_A for the Z_A -action on the symplectic vector space $H_A^1(\Sigma, \text{ad}(\xi))$. Theorem A makes precise the above remark that the singularities of $N(\xi)$ are “as simple as possible”. In fact, Theorem A reduces the study of the singularities of $N(\xi)$ to the standard example (2.4) on p. 52 of ARMS-GOTAY-JENNINGS [2]. Combining it with results of SJAMAAR-LERMAN [21] we shall show in [7] that the decomposition of $N(\xi)$ into connected components of orbit types of central Yang-Mills connections is a stratification in the strong sense in such a way that each stratum, being a smooth manifold, inherits a finite volume symplectic structure from the given data. This will in general refine the ATIYAH-BOTT-decomposition of the moduli space of *all* Yang-Mills connections. In particular, for $G = U(n)$, the unitary group, in the “coprime case”, cf. ATIYAH-BOTT [4], the component of the absolute minimum of the Yang-Mills functional has no singularities, that is, $N(\xi)$ is smooth, and our stratification then consists of a single piece.

The statement of Corollary B was obtained by ATIYAH-BOTT by means of symplectic reduction involving infinite dimensional spaces, see p. 587 of [4]. Our Corollary B avoids this infinite dimensional symplectic reduction, at the cost of exploiting the implicit function theorem for Banach manifolds. Another proof of the closedness of the symplectic structure at the non-singular points relying on finite dimensional techniques has recently been given by WEINSTEIN [22]. Corollary B also paves the way to handle arbitrary central Yang-Mills connections A , not just those yielding non-singular points of $N(\xi)$. The details are given in the follow up paper [11] where we construct a *stratified symplectic structure* in the sense of SJAMAAR [20] and SJAMAAR-LERMAN [21]; this is a Poisson structure defined at *every* point of $N(\xi)$; on each stratum, it restricts to the corresponding symplectic Poisson structure. As a stratified symplectic space, for every central Yang-Mills connection A , the space H_A will then be a local model for $N(\xi)$ near the point represented by A . Thus additional geometric information not spelled out here is lurking behind our Theorem A, cf. e. g. our paper [12].

In another follow up paper [8] we identify the strata of $N(\xi)$ with reductions to suitable subbundles. Thereby we *cannot* avoid running into principal bundles with *non-connected* structure groups, even when the structure group of the bundle ξ we started with is connected. This is the reason why the present theory has been set up for general compact not necessarily connected structure groups.

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1. Preliminaries

1.1. The space of connections as a Kähler manifold

Write $\Omega^* = \Omega^*(\Sigma, \text{ad}(\xi))$. The *data* we shall use throughout are the chosen orthogonal structure on g , the Riemannian metric on Σ , and a fixed orientation on Σ , with unique length one volume form vol_Σ in this orientation. We recall from [4] that the data induce

$$(1.1.1) \quad [\cdot, \cdot]: \Omega^* \otimes \Omega^* \longrightarrow \Omega^*, \quad \text{a graded Lie bracket;}$$

$$(1.1.2) \quad \wedge: \Omega^* \otimes \Omega^* \longrightarrow \Omega^*(\Sigma, \mathbf{R}), \quad \text{a graded commutative pairing;}$$

$$(1.1.3) \quad (\cdot, \cdot): \Omega^* \otimes \Omega^{2-*} \longrightarrow \mathbf{R}, \quad \text{a weakly non-degenerate bilinear pairing,}$$

given by $(\zeta, \lambda) = \int_\Sigma \zeta \wedge \lambda$;

$$(1.1.4) \quad \cdot: \Omega^* \otimes \Omega^* \longrightarrow \mathbf{R}, \quad \text{a *weak* inner product;}$$

$$(1.1.5) \quad *: \Omega^* \longrightarrow \Omega^{2-*}, \quad \text{a duality operator.}$$

In degree one, the pairing (1.1.3) and duality operator (1.1.5) amount to a *weakly* symplectic structure $\sigma = (\cdot, \cdot)$ and a complex structure $*$ on Ω^1 , respectively.

The space $\mathcal{A}(\xi)$ of connections on ξ is affine, having $\Omega^1(\Sigma, \text{ad}(\xi))$ as its group of translations, and hence the three pieces of structure $\sigma, *, \cdot$ extend to the space of connections $\mathcal{A}(\xi)$; moreover they fit together so that $\zeta \cdot \lambda = (\zeta, * \lambda)$ whence, in particular, they turn $\mathcal{A}(\xi)$ into a Kähler manifold in the appropriate sense; the Kähler structure is manifestly preserved by the action of the group $\mathcal{G}(\xi)$ of gauge transformations on $\mathcal{A}(\xi)$.

We recall from p. 546 of [4] that any three elements u, v, w in $\Omega^*(\Sigma, \text{ad}(\xi))$ satisfy

$$(1.1.6) \quad [u, v] \wedge w = u \wedge [v, w].$$

This implies that, for $|u| + |v| + |w| = 2$,

$$(1.1.7) \quad ([u, v], w) = (u, [v, w]).$$

Next, for every connection A , when $p + q = 1$, $\phi \in \Omega^p(\Sigma, \text{ad}(\xi))$ and $\psi \in \Omega^q(\Sigma, \text{ad}(\xi))$ satisfy

$$(1.1.8) \quad (\phi, d_A \psi) = (-1)^{|\phi|} (d_A \phi, \psi).$$

Finally, given α and β in $\Omega^1(\Sigma, \text{ad}(\xi))$, we have the identity

$$(1.1.9) \quad \alpha \wedge \beta = * \alpha \wedge * \beta.$$

1.2. Central connections

We denote the centre of G by Z , and we write z for its Lie algebra. It is a sub Lie algebra of the centre of g , stable under the adjoint representation. A *central* connection A on ξ is one whose curvature $K_A \in \Omega^2(\Sigma, \text{ad}(\xi))$ is a 2-form with values

in z . Thus in particular a flat connection is central. We recall that the topology of ξ determines an element $X_\xi \in z$ so that, given an arbitrary central Yang-Mills connection A , its curvature K_A equals the image $K_\xi \in \Omega^2(\Sigma, \text{ad}(\xi))$ of the constant 2-form $X_\xi \otimes \text{vol}_\Sigma \in \Omega^2(\Sigma, z)$ under the canonical map from $\Omega^2(\Sigma, z)$ to $\Omega^2(\Sigma, \text{ad}(\xi))$. We established this fact in (1.1) of our paper [9] for an arbitrary compact structure group, extending a result in Section 6 of ATIYAH-BOTT [4] for a connected structure group. Henceforth we denote by $d_A: \Omega^* \rightarrow \Omega^{*+1}$ the operator of covariant derivative.

Let now A be a central connection. It is clear that its operator of covariant derivative d_A is a differential, that is, satisfies $d_A d_A = 0$, since $d_A d_A = [K_A, \cdot]$. Hence the cohomology groups $H_A^* = H_A^*(\Sigma, \text{ad}(\xi))$ are defined. Since the operator of covariant derivative behaves as a derivation under both the graded Lie bracket (1.1.1) and the wedge product (1.1.2), (1.1.1) – (1.1.3) induce

$$(1.2.1) \quad [\cdot, \cdot]_A: H_A^* \otimes H_A^* \longrightarrow H_A^*, \quad \text{a graded Lie bracket,}$$

$$(1.2.2) \quad \wedge_A: H_A^* \otimes H_A^* \longrightarrow H^*(\Sigma, \mathbf{R}), \quad \text{a graded commutative pairing,}$$

$$(1.2.3) \quad (\cdot, \cdot)_A: H_A^* \otimes H_A^{2-*} \longrightarrow \mathbf{R}, \quad \text{a non-degenerate bilinear pairing,}$$

which, in particular, yields the symplectic structure $\sigma_A = (\cdot, \cdot)_A$ on H_A^1 mentioned already in the Introduction.

The kernel of the operator $d_A: \Omega^0(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^1(\Sigma, \text{ad}(\xi))$ of covariant derivative is the Lie algebra z_A of Z_A . Hence $H_A^0(\Sigma, \text{ad}(\xi))$ is the Lie algebra z_A of Z_A . Furthermore, (1.2.3) identifies $H_A^2(\Sigma, \text{ad}(\xi))$ with the dual z_A^* of z_A , and it is clear that the $\mathcal{G}(\xi)$ -action on $\mathcal{A}(\xi)$ induces an action of Z_A on $H_A^*(\Sigma, \text{ad}(\xi))$. Moreover, the corresponding infinitesimal z_A -action on $H_A^*(\Sigma, \text{ad}(\xi))$ is given by the restriction of the graded Lie bracket (1.2.1) to $H_A^0(\Sigma, \text{ad}(\xi)) \otimes H_A^*(\Sigma, \text{ad}(\xi))$, that is, by the assignment to $\phi \in H_A^0(\Sigma, \text{ad}(\xi))$ of the operation $X_\phi: H_A^* \rightarrow H_A^*$ given by

$$(1.2.4) \quad X_\phi(\eta) = [\phi, \eta]_A, \quad \eta \in H_A^*(\Sigma, \text{ad}(\xi)).$$

Since the $\mathcal{G}(\xi)$ -action on $\mathcal{A}(\xi)$ preserves σ , it is clear that the Z_A -action on $H_A^1(\Sigma, \text{ad}(\xi))$ preserves σ_A , and we have the following, the proof of which we leave to the reader, cf. [6].

Lemma 1.2.5. *For an arbitrary central connection A , the assignment to $\eta \in H_A^1(\Sigma, \text{ad}(\xi))$ of $\Theta_A(\eta) = \frac{1}{2}[\eta, \eta]_A$ yields a momentum mapping Θ_A from $H_A^1(\Sigma, \text{ad}(\xi))$ to z_A^* for the action of Z_A on the symplectic vector space $H_A^1(\Sigma, \text{ad}(\xi))$.*

1.3. Hodge decomposition

Let A be a connection on ξ . As usual we write $d_A^*: \Omega^* \rightarrow \Omega^{*-1}$ for the adjoint of d_A with respect to the weak inner product (1.1.4) on $\Omega^*(\Sigma, \text{ad}(\xi))$, cf. p. 552 of [4]. Since the inner product is only weak the existence of the adjoint relies on a suitable version of the Hodge decomposition theorem.

For $j = 1, 2$, we denote by $B_A^j(\Sigma, \text{ad}(\xi))$ the subspace $d_A(\Omega^{j-1}(\Sigma, \text{ad}(\xi)))$ of coboundaries in $\Omega^j(\Sigma, \text{ad}(\xi))$ and by \mathcal{P}_A the orthogonal projection from $\Omega^j(\Sigma, \text{ad}(\xi))$ to $B_A^j(\Sigma, \text{ad}(\xi))$ and, for $j = 0, 1, 2$, we denote by $\mathcal{H}_A^j(\Sigma, \text{ad}(\xi))$ the vector space of harmonic j -forms in $\Omega^j(\Sigma, \text{ad}(\xi))$. The Laplace operator

$$(1.3.1) \quad \Delta_A = d_A d_A^* + d_A^* d_A: \Omega^*(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^*(\Sigma, \text{ad}(\xi))$$

is manifestly Z_A -equivariant. We reproduce the following well known facts which rely on the properties of the corresponding *Green's* operator.

Proposition 1.3.2. *For a central connection A on ξ , for $j = 1, 2$, the restriction*

$$(1.3.3) \quad \Delta_A| = d_A d_A^*|: B_A^j(\Sigma, \text{ad}(\xi)) \rightarrow B_A^j(\Sigma, \text{ad}(\xi))$$

is an Z_A -equivariant isomorphism of real vector spaces.

Let A be a central connection on ξ . For $j = 0, 1, 2$, we write

$$(1.3.4) \quad \alpha_A: \Omega^j(\Sigma, \text{ad}(\xi)) \rightarrow \mathcal{H}_A^j(\Sigma, \text{ad}(\xi)), \quad \iota_A: \mathcal{H}_A^j(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^j(\Sigma, \text{ad}(\xi))$$

for the orthogonal projection and canonical injection, respectively. They are manifestly Z_A -equivariant. For $j = 1, 2$, we then consider the operator

$$(1.3.5) \quad h_A = d_A^* \Delta_A^{-1} \mathcal{P}_A: \Omega^j(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^{j-1}(\Sigma, \text{ad}(\xi)).$$

Clearly it also looks like $h_A = G_1 d_A^* \mathcal{P}_A = d_A^* G_2 \mathcal{P}_A$ where G_1 and G_2 refer to the corresponding *Green's* operators. We spell out some of its properties.

(1.3.6) *It is Z_A -equivariant.*

(1.3.7) *It satisfies $d_A^* h_A = 0$ and $*h_A = d_A * \Delta_A^{-1} \mathcal{P}_A$.*

(1.3.8) $\ker(h_A) = \ker(d_A^*)$.

(1.3.9) For $j = 1, 2$, we have $\mathcal{P}_A = d_A h_A: \Omega^j(\Sigma, \text{ad}(\xi)) \rightarrow B_A^j(\Sigma, \text{ad}(\xi))$.

The proofs of these properties are straightforward and left to the reader. Finally we spell out the following version of the *Hodge decomposition theorem*.

Lemma 1.3.10. *For a central connection A on ξ , the operators h_A furnish a chain homotopy $h_A: \text{Id} \simeq \iota_A \alpha_A$, that is, we have*

$$(1.3.11) \quad d_A h_A + h_A d_A = \text{Id} - \iota_A \alpha_A.$$

Let now A be a central connection. Then (1.3.10) implies that the obvious map

$$(1.3.12) \quad \kappa_A: \mathcal{H}_A^j(\Sigma, \text{ad}(\xi)) \rightarrow H_A^j(\Sigma, \text{ad}(\xi))$$

is an isomorphism of vector spaces; indeed, each cohomology class in $H_A^j(\Sigma, \text{ad}(\xi))$ has a unique harmonic representative. Furthermore, the duality operator (1.1.5) passes to a duality operator “ $*$ ” from $\mathcal{H}_A^*(\Sigma, \text{ad}(\xi))$ to $\mathcal{H}_A^{2-*}(\Sigma, \text{ad}(\xi))$, and the weak inner product (1.1.4) induces an inner product on each $\mathcal{H}_A^q(\Sigma, \text{ad}(\xi))$ which we denote by the same symbol “ \cdot ”; since these spaces are finite dimensional there is no difference here between “weak” and “strong”. It is clear that these pieces of structure together with the restriction of the symplectic structure σ constitute a hermitian structure on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$, having $*$ as its complex structure. By means of the isomorphism (1.3.12), we also have this structure on $H_A^1(\Sigma, \text{ad}(\xi))$; its symplectic structure is just σ_A . Furthermore, the Z_A -action on $H_A^1(\Sigma, \text{ad}(\xi))$ is in fact a unitary representation since it is compatible with all the structure, and the momentum mapping Θ_A is the unique one for this representation having the value zero at the origin.

2. The description of the singularities

In this Section we spell out and prove a more precise version of Theorem A in the Introduction. Following the arguments in [3] we use the Kuranishi theory of deformations to describe the exact structure of the singularities of our spaces of interest. Technically it may be worthwhile remembering that, for any connection A , the operator $d_A + d_A^*$ is elliptic, and it will be convenient to work with suitable Sobolev spaces.

The assignment to a connection A of its curvature K_A is a smooth map J from $\mathcal{A}(\xi)$ to $\Omega^2(\Sigma, \text{ad}(\xi))$. It is well known to be given by the formula

$$(2.1) \quad J(A + \eta) = K_{A+\eta} = K_A + d_A \eta + \frac{1}{2}[\eta, \eta], \quad \eta \in \Omega^1(\Sigma, \text{ad}(\xi)),$$

and its tangent map $dJ(A)$ at A amounts to the covariant derivative d_A from $\Omega^1(\Sigma, \text{ad}(\xi))$ to $\Omega^2(\Sigma, \text{ad}(\xi))$ where the tangent space $T_A \mathcal{A}(\xi)$ is identified with $\Omega^1(\Sigma, \text{ad}(\xi))$ as usual. Moreover, cf. what is said in (1.2) above, on the subspace $\mathcal{N}(\xi)$ of central Yang-Mills connections, the map J has a constant value $K_\xi \in \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$, determined by the topology of ξ .

Let $A \in \mathcal{N}(\xi)$ be a smooth central Yang-Mills connection, fixed throughout. Since A is a solution of the Yang-Mills equations (0.1) the value of its curvature $K_A = K_\xi$ lies in the space $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$ of harmonic 2-forms. Consider the Hodge decomposition

$$(2.2) \quad \Omega^2(\Sigma, \text{ad}(\xi)) = B_A^2(\Sigma, \text{ad}(\xi)) \oplus \mathcal{H}_A^2(\Sigma, \text{ad}(\xi)).$$

Clearly the point A of $\mathcal{N}(\xi)$ is *regular* for J if and only if d_A is surjective, that is, if and only if $H_A^2(\Sigma, \text{ad}(\xi))$ is zero. In a neighborhood of A , the pre-image $J^{-1}(K_\xi) = J^{-1}(0)$ is then a smooth manifold, and the tangent space to $\mathcal{N}(\xi)$ at A equals the space

$$(2.3) \quad T_A \mathcal{N}(\xi) = \ker(d_A) = Z_A^1(\Sigma, \text{ad}(\xi))$$

of 1-cocycles at A .

Suppose now that $H_A^2(\Sigma, \text{ad}(\xi))$ is non-zero. As before, let \mathcal{P}_A be the orthogonal projection from $\Omega^2(\Sigma, \text{ad}(\xi))$ onto $B_A^2(\Sigma, \text{ad}(\xi))$, and let

$$(2.4) \quad \mathcal{A}_A = (\mathcal{P}_A J)^{-1}(0) = J^{-1}(\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))).$$

Notice that $A + \eta \in \mathcal{A}_A$ if and only if $d_A^*(d_A \eta + \frac{1}{2}[\eta, \eta]) = 0$. By construction, the space \mathcal{A}_A contains $\mathcal{N}(\xi)$. Further, since the composite map $\mathcal{P}_A J$ from $\mathcal{A}(\xi)$ to $B_A^2(\Sigma, \text{ad}(\xi))$ is a submersion at A , in a neighborhood of A , \mathcal{A}_A is a smooth manifold in such a way that, for $A + \eta \in \mathcal{A}_A$,

$$T_{A+\eta} \mathcal{A}_A = \{\psi; \mathcal{P}_A d_{A+\eta} \psi = 0\} = \{\psi; d_{A+\eta} \psi \in \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))\} \subseteq \Omega^1(\Sigma, \text{ad}(\xi)).$$

In particular, $T_A \mathcal{A}_A = \ker(d_A) = Z_A^1(\Sigma, \text{ad}(\xi))$, and $Z_{A+\eta}^1(\Sigma, \text{ad}(\xi)) \subseteq T_{A+\eta} \mathcal{A}_A$. Define the smooth map

$$(2.5) \quad J^\sharp: \mathcal{A}_A \rightarrow \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$$

as the restriction of J so that, for $A + \eta \in \mathcal{A}_A$,

$$(2.6) \quad J^\sharp(A + \eta) = K_\xi + d_A \eta + \frac{1}{2}[\eta, \eta] \in \mathcal{H}_A^2(\Sigma, \text{ad}(\xi)).$$

Note that $J^\sharp(A) = K_\xi$ and that, for $A + \eta \in \mathcal{A}_A$, the tangent map $dJ^\sharp(A + \eta)$ from $T_{A+\eta}\mathcal{A}_A$ to $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$ is given by the restriction of the operator $d_{A+\eta}$; in particular, the derivative $dJ^\sharp(A): Z_A^1(\Sigma, \text{ad}(\xi)) \rightarrow \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$ is zero since so is the restriction of d_A to $Z_A^1(\Sigma, \text{ad}(\xi))$. It is clear that the space of central Yang-Mills connections $\mathcal{N}(\xi)$ is smooth near A and coincides with \mathcal{A}_A near A if and only if the map J^\sharp is constant, having constant value K_ξ . Hence:

Lemma 2.7. *For a 1-form $\eta \in \Omega^1(\Sigma, \text{ad}(\xi))$ having the property that $A + \eta \in \mathcal{A}_A$, the following are equivalent.*

- (1) *The connection $A + \eta \in \mathcal{A}_A$ is a central Yang-Mills connection;*
- (2) *$d_A \eta + \frac{1}{2}[\eta, \eta] = 0 \in \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$;*
- (3) *$[\eta, \eta]_A = 0 \in \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$. \square*

At this stage we thus obtain already the following well known.

Theorem 2.8. *The space $\mathcal{N}(\xi)$ of central Yang-Mills connections coincides with \mathcal{A}_A near A and hence is smooth near $A \in \mathcal{N}(\xi)$, having the space of 1-cocycles $Z_{A+\eta}^1(\Sigma, \text{ad}(\xi))$ as tangent space for every $A + \eta \in \mathcal{N}(\xi)$ near A , if and only if the symmetric bilinear pairing $[\cdot, \cdot]_A$ on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ is zero. \square*

Recall that after a choice $Q \in \Sigma$ of base point, the normal subgroup $\mathcal{G}^Q(\xi)$ of at Q based gauge transformations acts freely on $\mathcal{A}(\xi)$, and, after a choice $\widehat{Q} \in P$ of pre-image of Q has been made, evaluation of gauge transformations at Q furnishes a surjective homomorphism from $\mathcal{G}(\xi)$ onto G whose kernel equals $\mathcal{G}^Q(\xi)$. Consequently this surjection maps the stabilizer of an arbitrary connection isomorphically onto a closed subgroup of G . Under the present circumstances, the method of Kuranishi consists of parametrizing a neighborhood of A in \mathcal{A}_A equivariantly with respect to its stabilizer Z_A by a neighborhood of the tangent space of \mathcal{A}_A . Here are the details:

Recall that, in view of the Hodge decomposition theorem, an *infinitesimal* slice for the $\mathcal{G}(\xi)$ -action on $\mathcal{A}(\xi)$ is given by the *transverse gauge*, that is, by the affine subspace

$$\mathcal{S}_A = A + \ker(d_A^*: \Omega^1(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^0(\Sigma, \text{ad}(\xi))).$$

Standard analytic arguments involving Sobolev spaces then show that this infinitesimal slice generates a local slice but for the moment this is not important for us. We only note that $T_A \mathcal{S}_A = \ker(d_A^*)$ and that, with respect to (1.1.4), the Hodge decomposition

$$T_A \mathcal{A}(\xi) = d_A(\Omega^0(\Sigma, \text{ad}(\xi))) \oplus \ker(d_A^*)$$

is an orthogonal decomposition.

By means of the operator h_A , cf. (1.3.5), the corresponding *Kuranishi map* F_A from $\mathcal{A}(\xi)$ to itself is defined by

$$(2.9) \quad F_A(A + \eta) = A + \eta + \frac{1}{2}h_A[\eta, \eta], \quad \eta \in \Omega^1(\Sigma, \text{ad}(\xi)).$$

We shall use its properties spelled out below, cf. Lemmata 9 – 12 in [3].

(2.10) *It is Z_A -equivariant.*

(2.11) *It is a local diffeomorphism of a neighborhood of A to a neighborhood of A .*

(2.12) *It satisfies the formula*

$$d_A(F_A(A + \eta) - A) = \mathcal{P}_A(J(A + \eta)), \quad \eta \in \Omega^1(\Sigma, \text{ad}(\xi)).$$

(2.13) *Hence it identifies a neighborhood of A in \mathcal{A}_A with a neighborhood of A in $A + Z_A^1(\Sigma, \text{ad}(\xi))$.*

(2.14) *For every $\eta \in \Omega^1(\Sigma, \text{ad}(\xi))$, we have $d_A^*(F_A(A + \eta) - A) = d_A^*(\eta)$. Consequently F_A maps \mathcal{S}_A to itself in such a way that $F_A(A + \eta) \in \mathcal{S}_A$ implies $A + \eta \in \mathcal{S}_A$.*

Properties (2.13) and (2.14) above imply:

(2.15) *Near A , the intersection $\mathcal{A}_A \cap \mathcal{S}_A$ is a smooth finite dimensional manifold, and the Kuranishi map F_A restricts to a local diffeomorphism of $\mathcal{A}_A \cap \mathcal{S}_A$ onto $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$.*

Smoothness of $\mathcal{A}_A \cap \mathcal{S}_A$ near A follows also from the fact that \mathcal{A}_A and \mathcal{S}_A intersect transversely near A . With respect to (1.1.4), we now explicitly choose a ball \mathcal{B}_A around A in $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ in such a way that (i) *the space*

$$(2.16) \quad \mathcal{M}_A = F_A^{-1}(\mathcal{B}_A) \subseteq \mathcal{A}_A \cap \mathcal{S}_A,$$

is a smooth finite dimensional Z_A -manifold, and (ii) the Kuranishi map F_A restricts to a diffeomorphism

$$(2.17) \quad f_A: \mathcal{M}_A \rightarrow \mathcal{B}_A,$$

necessarily Z_A -equivariant. In fact, the space $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ being of finite dimension, the restriction of the inner product (1.1.4) yields a norm on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ in the usual (strong) sense. Moreover, the action of the group $\mathcal{G}(\xi)$ of gauge transformations restricts to an action of Z_A on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ and hence on the affine subspace $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$, and the weak inner product (1.1.4) is invariant under gauge transformations whence any ball therein is an Z_A -invariant subspace; in particular, the ball \mathcal{B}_A in $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ is Z_A -invariant. Since F_A is Z_A -equivariant, \mathcal{M}_A inherits a Z_A -action, and f_A is Z_A -equivariant.

We now have the machinery in place to show that the diffeomorphism (2.17) yields a symplectic change of coordinates: Let ω_A be the restriction of the (weakly) symplectic structure σ on $\mathcal{A}(\xi)$ to the smooth submanifold \mathcal{M}_A ; it is necessarily closed.

Lemma 2.18. *The diffeomorphism f_A is compatible with the 2-forms ω_A and σ in the sense that, for $A + \eta \in \mathcal{M}_A$, given $\psi, \vartheta \in T_{A+\eta}\mathcal{M}_A$, we have*

$$\sigma(\psi, \vartheta) = \omega_A(\psi, \vartheta) = \sigma(f'_A(A + \eta)\psi, f'_A(A + \eta)\vartheta).$$

Consequently ω_A is non-degenerate, that is, σ passes to a symplectic structure on \mathcal{M}_A .

Proof. Let $A + \eta \in \mathcal{M}_A$. In a neighborhood of $A + \eta$, the map f_A looks like

$$f_A(A + \eta + \psi) = A + \eta + \frac{1}{2}h_A[\eta, \eta] + (\psi + h_A[\eta, \psi]) + \frac{1}{2}h_A[\psi, \psi],$$

for suitable $\psi \in \Omega^1(\Sigma, \text{ad}(\xi))$, cf. (2.9). Consequently its derivative

$$f'_A(A + \eta): T_{A+\eta}\mathcal{M}_A \rightarrow T_{F_A(A+\eta)}\mathcal{H}_A^1(\Sigma, \text{ad}(\xi)) = \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$$

at $A + \eta \in \mathcal{M}_A$ is given by the assignment to $\psi \in T_{A+\eta}\mathcal{M}_A$ of $\psi + h_A[\eta, \psi]$. Thus the statement of the Lemma will be a consequence of the following.

Proposition 2.19. *Given $\psi, \vartheta \in T_{A+\eta}\mathcal{S}_A = \ker(d_A^*: \Omega^1(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^0(\Sigma, \text{ad}(\xi)))$, we have*

$$\sigma(\psi, \vartheta) = \sigma(\psi + h_A[\eta, \psi], \vartheta + h_A[\eta, \vartheta]).$$

Proof. Clearly it will suffice to show that $\sigma(h_A[\eta, \psi], \vartheta)$, $\sigma(\psi, h_A[\eta, \vartheta])$, and $\sigma(h_A[\eta, \psi], h_A[\eta, \vartheta])$ are zero. In order to see this we recall that the duality operator $*$ is symplectic whence $\sigma(u, v) = \sigma(*u, *v)$, whatever $u, v \in \Omega^1(\Sigma, \text{ad}(\xi))$. By virtue of (1.3.7), letting $\tau = *\Delta_A^{-1}\mathcal{P}_A[\eta, \psi] \in \Omega^0(\Sigma, \text{ad}(\xi))$, we have, cf. (1.1.8),

$$\sigma(*h_A[\eta, \psi], *\vartheta) = (d_A\tau, *\vartheta) = (\tau, d_A*\vartheta) = 0$$

whence

$$\sigma(h_A[\eta, \psi], \vartheta) = \sigma(*h_A[\eta, \psi], *\vartheta) = 0.$$

The same kind of argument shows that $\sigma(\psi, h_A[\eta, \vartheta])$ is zero. Finally, to see that $\sigma(*h_A[\eta, \psi], *h_A[\eta, \vartheta])$ is zero, we proceed as above and obtain

$$\sigma(*h_A[\eta, \psi], *h_A[\eta, \vartheta]) = (\tau, d_A*h_A[\eta, \vartheta]).$$

However this is zero since d_A*h_A is zero, cf. (1.3.7). \square

Corollary 2.20. *The local diffeomorphism f_A from \mathcal{M}_A to $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ is a symplectic change of coordinates, that is, it yields Darboux coordinates on \mathcal{M}_A . \square*

Our next aim is to examine the restriction J_A of J^\sharp to \mathcal{M}_A , cf. (2.5). We recall that (1.2.3) identifies $H_A^2(\Sigma, \text{ad}(\xi))$ with the dual z_A^* of z_A and, cf. [4], that J , cf. (2.1), is a momentum mapping for σ and the $\mathcal{G}(\xi)$ -action on $\mathcal{A}(\xi)$, the space $\Omega^2(\Sigma, \text{ad}(\xi))$ being identified with the dual of the Lie algebra $\mathfrak{g}(\xi) = \Omega^0(\Sigma, \text{ad}(\xi))$ of infinitesimal gauge transformations via (1.1.3). This implies at once the following.

Lemma 2.21. *The composition of J_A with the canonical isomorphism κ_A from $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$ to $H_A^2(\Sigma, \text{ad}(\xi))$, cf. (1.3.12), is a momentum mapping for the action of Z_A on the symplectic manifold \mathcal{M}_A . \square*

For $\phi \in z_A = H_A^0(\Sigma, \text{ad}(\xi))$, let X_ϕ denote the vector field on $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ induced by the infinitesimal z_A -action so that, cf. (1.2.4), for $\eta \in \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$,

$$(2.22) \quad X_\phi(A + \eta) = [\phi, \eta] \in \mathcal{H}_A^1(\Sigma, \text{ad}(\xi)) = T_{A+\eta}(A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))).$$

Lemma 2.23. *The assignment to $A + \eta \in A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ of*

$$j_A(A + \eta) = \kappa_A(K_\xi) + \frac{1}{2}[\eta, \eta]_A$$

yields a momentum mapping j_A for the action of Z_A on the affine symplectic space $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$, in fact, the unique one having the value $\kappa_A(K_\xi)$ at the point A .

Proof. This is established in much the same way as (1.2.5), by means of the canonical isomorphism (1.3.12), combined with the canonical symplectomorphism of affine symplectic manifolds from $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ to $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$. \square

As far as the statement of (2.23) is concerned, the term involving K_ξ may safely be ignored. It has been included to have a consistent result in (2.24) below.

Theorem 2.24. *The symplectomorphism f_A preserves the momentum mappings in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{M}_A & \xrightarrow{J_A} & \mathcal{H}_A^2(\Sigma, \text{ad}(\xi)) \\ f_A \downarrow & & \downarrow \kappa_A \\ A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi)) & \xrightarrow{j_A} & \mathcal{H}_A^2(\Sigma, \text{ad}(\xi)) \end{array}$$

is commutative where κ_A refers to the canonical isomorphism from $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$ to $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$, cf. (1.3.12).

In order to prove this we need some preparations.

Lemma 2.25. *For $A + \eta \in \mathcal{A}_A \cap \mathcal{S}_A$, the elements $[\eta, h_A[\eta, \eta]]$ and $[h_A[\eta, \eta], h_A[\eta, \eta]]$ are coboundaries, that is, pass to zero in $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$.*

To see this we proceed as follows: Since $A + \eta \in \mathcal{A}_A$ the element $d_A\eta + \frac{1}{2}[\eta, \eta]$ lies in $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$, and hence $h_A d_A\eta + \frac{1}{2}h_A[\eta, \eta] = 0$, since $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi)) = \ker(h_A)$, cf. (1.3.8). Hence it will suffice to show that $[\eta, h_A d_A\eta]$ and $[h_A d_A\eta, h_A d_A\eta]$ pass to zero in $\mathcal{H}_A^2(\Sigma, \text{ad}(\xi))$.

Lemma 2.26. *For $\phi \in \mathcal{H}_A^0(\Sigma, \text{ad}(\xi))$ and $\eta \in \Omega^1(\Sigma, \text{ad}(\xi))$, $[\phi, * \eta]$ equals $*[\phi, \eta]$.*

Proof. The action of the group $\mathcal{G}(\xi)$ of gauge transformations on $\mathcal{A}(\xi)$ preserves $*$ in the sense that, given a gauge transformation γ , we have

$$(T_A\gamma) \circ * = * \circ (T_A\gamma): T_A\mathcal{A}(\xi) = \Omega^1(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^1(\Sigma, \text{ad}(\xi)) = T_{\gamma A}\mathcal{A}(\xi).$$

Consequently the action of Z_A on $\Omega^1(\Sigma, \text{ad}(\xi)) = T_A\mathcal{A}(\xi)$ preserves $*$. Given $\psi \in \Omega^0(\Sigma, \text{ad}(\xi)) = \mathfrak{g}(\xi)$, let $X_\psi: \mathcal{A}(\xi) \rightarrow \Omega^1(\Sigma, \text{ad}(\xi))$ be the vector field on $\mathcal{A}(\xi)$ coming from the infinitesimal $\mathfrak{g}(\xi)$ -action on $\mathcal{A}(\xi)$; for a connection \tilde{A} , it is given by $X_\psi(\tilde{A}) = -d_{\tilde{A}}(\psi)$, and its derivative $dX_\psi(\tilde{A})$ looks like

$$dX_\psi(\tilde{A}): \Omega^1(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^1(\Sigma, \text{ad}(\xi)).$$

Since the action of the isotropy subgroup Z_A on $\Omega^1(\Sigma, \text{ad}(\xi)) = T_A\mathcal{A}(\xi)$ preserves the duality operator $*$, for $\phi \in \mathcal{H}_A^0(\Sigma, \text{ad}(\xi)) = z_A$, we have

$$(T*) \circ dX_\phi(A) = dX_\phi(A) \circ *.$$

See Lemma 4 in [3] and Proposition 4.1.28 in [1] for details. Since the range of X_ϕ is a linear space, $T*$ amounts to $*$. However, $d_{A+\eta}\psi = d_A\psi + [\eta, \psi]$, whence $-dX_\psi(A)(\eta) = [\psi, \eta]$. Consequently for $\phi \in \mathcal{H}_A^0(\Sigma, \text{ad}(\xi)) = z_A$, we get

$$[\phi, * \eta] = -dX_\phi(A)(* \eta) = - * dX_\phi(A)(\eta) = *[\phi, \eta]$$

as asserted. \square

Corollary 2.27. *For $\phi \in H_A^0(\Sigma, \text{ad}(\xi))$ and $\eta, \vartheta \in \Omega^1(\Sigma, \text{ad}(\xi))$, as real valued 2-forms,*

$$(2.27.1) \quad [\eta, \vartheta] \wedge \phi = [* \eta, * \vartheta] \wedge \phi.$$

Consequently, given $\eta, \vartheta \in \Omega^1(\Sigma, \text{ad}(\xi))$, the 2-forms $[\eta, \vartheta]$ and $[\eta, \vartheta]$ represent the same class in $H_A^2(\Sigma, \text{ad}(\xi))$.

Proof. Applying the identity (1.1.9) with $\alpha = \eta$ and $\beta = [\vartheta, \phi]$ and keeping in mind that, in view of (2.26), $*[\vartheta, \phi] = [* \vartheta, \phi]$, we obtain, cf. (1.1.6),

$$[\eta, \vartheta] \wedge \phi = \eta \wedge [\vartheta, \phi] = * \eta \wedge *[\vartheta, \phi] = * \eta \wedge [* \vartheta, \phi] = [* \eta, * \vartheta] \wedge \phi.$$

To verify the other statement, let $\phi \in H_A^0(\Sigma, \text{ad}(\xi))$. With reference to (1.2.3), in view of (2.27.1), we then have

$$(\phi, [\eta, \vartheta]_A)_A = (\phi, [* \eta, * \vartheta]_A)_A.$$

This implies the assertion, since the bilinear pairing (1.2.3) is non-degenerate. \square

We can now complete the proof of Lemma 2.25. In view of (2.27) it will suffice to show that $[\eta, * h_A d_A \eta]$ and $[* h_A d_A \eta, * h_A d_A \eta]$ pass to zero in $H_A^2(\Sigma, \text{ad}(\xi))$. However, cf. (1.3.7),

$$* h_A d_A \eta = d_A * \Delta_A^{-1} d_A \eta = d_A \psi$$

where $\psi = * \Delta_A^{-1} d_A \eta \in \Omega^0(\Sigma, \text{ad}(\xi))$. Consequently

$$[* \eta, * h_A d_A \eta] = [* \eta, d_A \psi] = d_A [* \eta, \psi] - [d_A * \eta, \psi].$$

However, since also $\eta \in \mathcal{S}_A$, $d_A^* \eta = 0$ whence $d_A * \eta = 0$, that is, $[\eta, * h_A d_A \eta]$ equals $d_A [* \eta, \psi]$. Likewise,

$$[* h_A d_A \eta, * h_A d_A \eta] = [d_A \psi, d_A \psi] = d_A [d_A \psi, \psi].$$

Consequently $[\eta, * h_A d_A \eta]$ and $[* h_A d_A \eta, * h_A d_A \eta]$ both pass to zero in $H_A^2(\Sigma, \text{ad}(\xi))$. This completes the proof of (2.25). \square

Proof of (2.24). Let $A + \eta \in \mathcal{M}_A$. In view of (2.25), the elements $[h_A[\eta, \eta], \eta]$ and $[h_A[\eta, \eta], h_A[\eta, \eta]]$ pass to zero in $H_A^2(\Sigma, \text{ad}(\xi))$. Consequently we have

$$\begin{aligned} j_A F_A(A + \eta) &= j_A \left(A + \eta + \frac{1}{2} h_A[\eta, \eta] \right) \\ &= K_\xi + \frac{1}{2} \left[\eta + \frac{1}{2} h_A[\eta, \eta], \eta + \frac{1}{2} h_A[\eta, \eta] \right]_A \\ &= K_\xi + \frac{1}{2} [\eta, \eta]_A = \kappa_A f_A(A + \eta). \quad \square \end{aligned}$$

Henceforth we write $\mathcal{N}_A = \mathcal{N}(\xi) \cap \mathcal{M}_A$.

Corollary 2.28. *The symplectomorphism f_A maps \mathcal{N}_A locally 1-1 onto the cone*

$$\mathcal{C}_A = A + \{\eta \in \mathcal{H}_A^1(\Sigma, \text{ad}(\xi)); [\eta, \eta]_A = 0 \in \mathcal{H}_A^2(\Sigma, \text{ad}(\xi))\}.$$

Finally, let

$$(2.29) \quad \Phi_A: \mathcal{M}_A \rightarrow \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$$

be the smooth map defined by

$$\Phi_A(A + \eta) = \kappa_A(F_A(A + \eta) - A) = \kappa_A(\eta + \frac{1}{2}h_A[\eta, \eta]), \quad A + \eta \in \mathcal{M}_A,$$

where κ_A refers to the isomorphism (1.3.12). In view of (2.15), this is an injective immersion, in fact, cf. (2.20), a local symplectomorphism. Moreover, let

$$(2.30) \quad \vartheta_A: \mathcal{M}_A \rightarrow \mathcal{H}_A^2(\Sigma, \text{ad}(\xi)) = z_A^*$$

be the smooth map defined by $\vartheta_A(x) = \kappa_A J_A(x) - \kappa_A(K_\xi)$, for $x \in \mathcal{M}_A$. Since K_ξ remains invariant under the Z_A -action, Lemma 2.21 implies that ϑ_A is a momentum mapping for the action of Z_A on the symplectic manifold \mathcal{M}_A ; in fact, it is the unique one having the value zero at the point A .

Corollary 2.31. *The local diffeomorphism Φ_A maps \mathcal{M}_A locally 1-1 symplectically and Z_A -equivariantly onto $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ and, furthermore, preserves the momentum mappings in the sense that $\vartheta_A = \Theta_A \Phi_A: \mathcal{M}_A \rightarrow z_A^*$.*

From this we deduce a more precise version of Theorem A in the Introduction.

Theorem 2.32. *The local symplectomorphism Φ_A induces a homeomorphism of a neighborhood of $[A]$ in $N(\xi)$ onto a neighborhood of zero in the Marsden-Weinstein reduced space $\mathcal{H}_A = \Theta_A^{-1}(0)/Z_A$.*

Proof. Since Z_A is a compact group, a suitable Sobolev completion of the space $\ker(d_A^*)$ may be endowed with a Z_A -invariant inner product. Any ball with respect to this inner product will then inherit a Z_A -action. By the slice theorem, cf. e. g. [17], [5], the map from \mathcal{M}_A/Z_A to $\mathcal{A}(\xi)/\mathcal{G}(\xi)$ induced by the injection of \mathcal{M}_A into $\mathcal{A}(\xi)$ is itself injective provided the ball \mathcal{B}_A around $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ coming into play in (2.16) is chosen sufficiently small; this map restricts to a homeomorphism of $N_A = \mathcal{N}_A/Z_A$ onto a neighborhood of $[A]$ in $N(\xi)$. On the other hand, in view of (2.31), the local symplectomorphism Φ_A identifies N_A locally 1-1 with \mathcal{H}_A . \square

We shall say that a central Yang-Mills connection A is *non-singular* if its stabilizer Z_A acts trivially on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$; the point $[A]$ of $N(\xi)$ will then be said to be *non-singular*.

Theorem 2.33. *Near a non-singular point $[A]$ the space $N(\xi)$ is smooth.*

Proof. In fact, for a non-singular central Yang-Mills connection A the momentum mapping Θ_A is zero and \mathcal{H}_A coincides with $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$. \square

It may happen that the subspace of smooth points of $N(\xi)$ is larger than that of its non-singular ones; for example this occurs for $G = \text{SU}(2)$ over a surface of genus 2, see [12]. However the subspace of non-singular points is exactly that where the symplectic structure is defined, that is, the symplectic structure on the subspace of non-singular points cannot be extended to other smooth points.

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